Parameter Identification in Elliptic Variational Inequalites

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This contribution is based on [3, 4, 5] and is concerned with the inverse problem of parameter identification in elliptic variational inequalities (VIs) of the first and second kind.

A prominent example of the latter class is the following *direct* problem: Find the function u in $H^1(\Omega) = \{v \in L^2(\Omega) : \nabla v \in (L^2(\Omega))^d\}$ on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ such that for any $v \in H^1(\Omega)$ there holds

$$\int_\Omega e(x)\ [\nabla u \cdot \nabla(v-u) + u(v-u)] + \int_{\partial\Omega} f(s)\ |u|(|v| - |u|) \geq \int_\Omega g(x)\ (v-u)\,. \quad (1)$$

This VI (1) provides a simplified scalar model of the Tresca frictional contact problem in linear elasticity. By the classic theory of variational inequalities there is a unique solution u of (1) if the datum g that enters the right-hand side is given in $L^2(\Omega)$ and moreover, the "ellipticity" parameter $e > 0$ in $L^\infty(\Omega)$ and the "friction" parameter $f > 0$ in $L^\infty(\partial\Omega)$ are known. Here we study the *inverse* problem that asks for the distributed parameters e and f, when the state u or, what is more realistic, some approximation $\tilde{u}$ from measurement is known. In other words, we are interested in the variable parameters e and f such that $u(e, f) = \tilde{u}$, what is however unrealistic due to the lack of regularity in the measured data. Consequently, the inverse problem of parameter identification will be posed as an optimization problem which aims to minimize the misfit function, namely the gap between the solution $u = u(e, f)$ and the measured data $\tilde{u}$. This approach has been precisely the case with simpler inverse problems. To the best of our knowledge, this is the first work on the inverse problem of parameter identification in variational inequalities that does not only treat the parameter e that is linked to a bilinear form here (or a linear form), but also the parameter f linked to a nonlinear nonsmooth function, like the modulus function above.

To cover this frictional contact model problem as well as other non-smooth problems from continuum mechanics we propose the following new abstract framework. Let (as above) V be a Hilbert space; moreover E, F Banach spaces with convex closed cones $E_+ \subset E$ and $F_+ \subset F$. Let as with [1], $t : E \times V \times V \to \mathbb{R}, (e, u, v) \mapsto t(e, u, v)$

a trilinear form and $l : V \to \mathbb{R}, v \mapsto l(v)$ a linear form. Assume that t is continuous such that $t(e, \cdot, \cdot)$ is $V-$elliptic for any fixed $e \in \text{ int } E_+$. Now in addition we have a "semisublinear form" $s : F \times V \to \mathbb{R}, (f, u) \mapsto s(f, u)$, that is, for any $u \in V$, $s(f, u)$ is linear in its first argument f on F and for any $f \in F_+$, $s(f, \cdot)$ is sublinear, continuous, and nonnegative on V. Moreover assume that $s(f, 0_V) = 0$ for any $f \in F$.

Then the forward problem is the following VI: Given $e \in \text{ int } E_+$ and $f \in F_+$, find $u \in V$ such that

$$t(e; u, v-u) \ + \ s(f; v) \ - \ s(f; u) \ \geq \ l(v-u) \,, \forall v \in V \,. \tag{2}$$

Now with given convex closed subsets $E^{\text{ad}} \subset \text{ int } E_+$ and $F^{\text{ad}} \subset F_+$ we seek to identify two parameters, namely the "ellipticity" parameter e in E^{ad} and the "friction" parameter f in F^{ad}.

In the first step we investigate the dependence of the solution of the forward problem on these parameters. We assume

$$t(e; u, v) \leq \bar{t} \ ||e||_E \ ||u||_V \ ||v||_V, \forall e \in E, \ u \in V, \ v \in V \tag{3}$$

$$t(e; u, u) \geq \underline{t} \ ||u||_V^2, \forall e \in E^{\text{ad}} \subset E, \ u \in V \tag{4}$$

$$|s(f; u_2) - s(f; u_1)| \leq \bar{s} \|f\|_F \|u_2 - u_1\|_V, \forall f \in F; u_1 \in V, u_2 \in V \,. \tag{5}$$

These assumptions can be verified in the model problem and further applications. Under these assumptions we obtain the following Lipschitz continuity result.

Theorem. Consider the uniquely defined solution map $(e, f) \in E^{\text{ad}} \times F^{\text{ad}} \mapsto u = S(e, f)$. Let $e_i \in E^{\text{ad}}, f_i \in F^{\text{ad}}$ $(i = 1, 2)$. Then there holds for some constant $c > 0$

$$\|S(e_2, f_2) - S(e_1, f_1)\|_V \leq c \, \{\|e_1 - e_2\|_E \ + \ \|f_1 - f_2\|_F\} \,.$$

Then we present an optimization approach to the parameter identification problem as follows. Let an observation $\tilde{u} \in V$ be given. Then the parameter identification problem studied in this paper reads: Find parameters $e \in E^{\text{ad}}, f \in F^{\text{ad}}$ such that $u = S(e, f)$ minimizes the "misfit function" $j(e, f) := \frac{1}{2} \ \|S(e, f) - \tilde{u}\|^2$. Here we assume that the sought ellipticity and friction parameters are smooth enough to satisfy with compact imbeddings $E^{\text{ad}} \subset \hat{E} \subset\subset E; F^{\text{ad}} \subset \hat{F} \subset\subset F$. Thus with given weights $\beta > 0, \gamma > 0$ we pose the stabilized optimization problem

$$(OP) \qquad \text{minimize } j(e, f) + \frac{\beta}{2} \ \|e\|_{\hat{E}}^2 + \frac{\gamma}{2} \|f\|_{\hat{F}}^2$$
$$\text{subject to } e \in E^{\text{ad}}, \ f \in F^{\text{ad}}$$

Theorem. Suppose the above compact imbeddings. Suppose that the trilinear form t satisfies (3) and (4) and that the semisublinear form s satisfies (5). Then (OP) admits an optimal (not necessarily unique!) solution $(e^*, f^*, u) \in E^{\text{ad}} \times F^{\text{ad}} \times V$, where $u = S(e^*, f^*)$, i.e. $u \in V$ solves the VI (2).

Finally we turn to finite dimensional approximation in the optimization approach and establish a convergence result employing the Galerkin method and Mosco set convergence what can be realized by finite element, respectively boundary element methods; see e.g. [6, 2] for the forward problem.

References

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